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Quantum coherent tunneling between two atomic-molecular Bose-Einstein condensates

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Abstract. We study the quantum coherent tunneling dynamics of two weakly coupled atomic-molecular Bose-Einstein condensates (AMBEC). A weak link is supposed to be provided by a double-well trap. The regions of parameters where the macroscopic quantum localization of the relative atomic population occurs are revealed. The different dynamical regimes are found depending on the value of nonlinearity, namely, coupled oscillations of population imbalance of atomic and molecular condensate, including irregular oscillations regions, and macroscopic quantum self trapping regimes. Quantum means and quadrature variances are calculated for population of atomic and molecular condensates and the possibility of quadrature squeezing is shown via stochastic simulations within P-positive phase space representation method. Linear tunnel coupling between two AMBEC leads to correlations in quantum statistics.

PACS. 03.75.-b Matter waves – 03.75.Gg Entanglement and decoherence in Bose-Einstein condensates – 03.75.Lm Tunneling, Josephson effect, Bose-Einstein condensates in periodic potentials, solitons, vortices and topological excitations – 05.30.Jp Boson systems

1 Introduction

In the last few years a sharp increase of interest in the dynamics of Bose-Einstein condensates (BEC) has emerged as a consequence of experimental achievements that proved it possible to create a BEC trapped by magnetic fields [1–3]. This initially concerned mainly the atomic BECs. More recently, it was found that there exists the possibility to create atomic-molecular Bose-Einstein condensates (AMBEC). It was shown theoretically [4–6] that an atomic condensate can be converted coherently to a condensate of diatomic molecules. Recently some experimental data was obtained showing the possibility to create mixing atoms and molecules in condensate [8]. Coherent coupling between atoms and molecules in BEC and oscillations between atomic and molecular states has been observed recently in [9]. Two different physical approaches were suggested to create Bose-Einstein condensate consisting of resonantly coupled atoms and molecules. In the first a resonance coupling could be achieved by application of external magnetic field with strength tuned close to Feshbach resonance [4,5]. Also, it was suggested to use direct photoassosiation process, in which the atomic condensate is coupled to the diatomic molecular condensate through two-photon Raman transition [6,8]. The equations describing both processes have the same mathematical structure and in addition have the direct analogue with the second harmonic generation equations in nonlinear optics. AMBEC formation, is believed, could be the way for development of a molecular laser and for coherent stimulation of chemical reactions [6], so-called superchemistry.

Several interesting effects have been predicted, using formal analogue with the second harmonic generation, for the dynamics of the mixed AMBEC, among them Josephson-like oscillations between atomic and molecular condensates [4–6] and the formation of stable soliton-like trapped states [6,10]. Also different physical mechanisms were suggested to convert most effectively the atomic condensate to molecular one using two-color Raman photoassociation scheme [12], or combination of stimulated Raman transition with time-dependent magnetic field near Feshbach resonance [13]. The properties of quantum AMBEC is another related topic attracting much interest. In [14] the optical parametric oscillator analogue for atom-molecular condensate, or the photodissociation process of initially molecular Bose condensate to atomic one have been investigated and it was shown that the atomic condensate appears in squeezed state. Dynamical quantum

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effects in AMBEC have been studied in [15,16]. Some interesting effects have been found which do not have analogies in semiclassical treatment. Also, decoherence effects in AMBEC induced by noncondenced atoms or by the mutual correlations have been considered [17–19].

On the other hand, the existence of macroscopic quantum phase difference in processes connected with the atomic waves has been shown to exist. This effect is observed in a system consisting of two tunnel-coupled atomic Bose-Einstein condensates [20–22,27,29–31,33]. The trap potential has the double-well structure. This is in direct analogy to the ac Josephson effect [23] and the periodic exchange of power and switching of electromagnetic waves between cores in nonlinear optical couplers [24]. In particular, there is a direct analogy for tunneling phenomena between two alongated Bose-Einstein condensates and two tunnel-coupled single mode optical fibers. In this optical analogy the role of chemical potential is played by the propagation constant, and the nonlinear interaction between atoms is the analog of the Kerr nonlinearity in optical media. The tunnel-coupling occurs from the overlaps of the electromagnetic fields outside of dielectric cylinders (fibers) [25,26] and it exactly corresponds to the tunnel coupling between two Bose-Einstein condensates, spatially separated in a double well trap, due to the overlaps of the wavefunctions.

A natural step in the investigation of nonlinearly coupled BECs is the consideration of tunneling phenomena in interacting atomic-molecular condensates. In this respect, it should be noted that in such situation we couple two different nonlinear oscillatory processes. As will be shown in the present paper, this entails new and complex dynamical regimes. We will consider problem in two aspects. First we analyze the problem in four modes approximation. In spite the well known difficulties which meets such approximation for present experimental status, it may be useful for investigation of the threshold phenomena in atomic condensate with taking into account the small molecular subsystem. This model is interesting also for nonlinear optics of quadratic nonlinear couplers where only few numerical results are known for the switching and power exchange phenomena [32]. To go beyond of the four modes model we will consider the influence of quantum fluctuations to dynamics of coupled AMBEC's and its quantum statistical properties. It is important and new issue, which needs a consideration.

The outline of the paper is as follows: in Section 2 we formulate the problem and introduce main equations; in Section 3, the classical Gross-Pitaevskii equations will be solved numerically in the four-mode approximation, and different dynamical regions will be characterized; in Section 4 we analyze the role of quantum fluctuations using P-positive phase space method and stochastic simulations to study the quantum mean dynamics and quantum statistics of the coupled AMBEC. Section 5 is devoted to conclusions.

2 Formulation of problem

The second-quantized Hamiltonian $\hat{H}$ for AMBEC in a trap can be written in the following form [4]

$$\begin{aligned}\hat{H} = &\int d\vec{r}\hat{\psi}_a^+(\vec{r})\left[-\frac{\hbar^2}{2M}\nabla^2 + V(\vec{r})\right]\hat{\psi}_a(\vec{r}) \\ &+ \frac{\lambda_a'}{2}\int d\vec{r}\hat{\psi}_a^+(\vec{r})\hat{\psi}_a^+(\vec{r})\hat{\psi}_a(\vec{r})\hat{\psi}_a(\vec{r}) \\ &+ \int d\vec{r}\hat{\psi}_m^+(\vec{r})\left[-\frac{\hbar^2}{4M}\nabla^2 + V(\vec{r})\right]\hat{\psi}_m(\vec{r}) \\ &+ \lambda'\int d\vec{r}\left[\hat{\psi}_a^2(\vec{r})\hat{\psi}_m^+(\vec{r}) + \hat{\psi}_a^{+2}(\vec{r})\hat{\psi}_m(\vec{r})\right], \quad (1)\end{aligned}$$

where $\hat{\psi}_a$ and $\hat{\psi}_m$ are the field operators of atoms and quasibound molecules, respectively. $V(\vec{r})$ represents the trapping potential, $\lambda_a' = 4\pi a\hbar^2/(2M)$ with M being the atomic mass and a the scattering length, λ' denotes the coupling constant of atom-molecule interaction, and detuning is assumed to be zero. $\hbar$ is the Planck constant. The atom-molecule and molecule-molecule interactions are not included as their parameters in the low-energy regime are unknown.

One measurement for λ_a' has recently been performed and agreed with value $\lambda_a' \approx -6.9 \times 10^{-51}$ Jm^3 [8]. For χ we have $\lambda' \approx 7.4 \times 10^{-41}$ Jm^3. By Feshbach resonances one can increase atom-atom interactions and so neglect the molecular-molecular interactions.

Note that the Hamiltonian is an effective Hamiltonian. The replacement of the interatomic potential by a δ-function pseudopotential leads to the ultraviolet divergences of the observable quantities. So it is necessary to use ultraviolet momentum cutoff with $k_m \sim 1/a$, where a is the longest scattering length (typically $k_m \sim 1$ nm^{-1}) (see [6] for more details). If we use the bare values λ_a', λ' it is required to introduce the momentum cutoff k_m. Returning to the renormalized values of λ_a', λ' we obtain the mean field equations of the effective field theory with complex coefficients of linear and nonlinear terms in GP equations (see the discussion in the recent review [7]). This effect leads to the damped oscillations and as the result to the shift in the threshold between the macroscopic quantum tunneling and the Josephson oscillations regimes. This problem requires the separate consideration.

In the mean field approximation one neglects the quantum and thermal fluctuations and replaces in the Heisenberg equations of motion the operators $\hat{\psi}_a$ and $\hat{\psi}_m$ by their mean values Ψ_a and Ψ_m. Then the Gross-Pitaevskii equations for the AMBEC in the trap have the form [6]

$$i\hbar\frac{\partial\Psi_a}{\partial t} = -\frac{\hbar^2}{2M}\Delta\Psi_a + V(\vec{r})\Psi_a + 2\lambda'\Psi_a^*\Psi_m + \lambda_a'|\Psi_a|^2\Psi_m, \quad (2)$$

$$i\hbar\frac{\partial\Psi_m}{\partial t} = -\frac{\hbar^2}{4M}\Delta\Psi_m + V(\vec{r})\Psi_m + \lambda'\Psi_a^2. \quad (3)$$

The generalized number of atoms is conserved

$$N = \int dV(|\Psi_a|^2 + 2|\Psi_m|^2), \tag{4}$$

with each molecule counting as two atoms.

Let us consider a well separated AMBEC in double-well trap. Then the condensates wavefunctions overlap only in the region of exponentially small tails and the tunnel coupling is small. In this situation, the solution can be presented in the form

$$\begin{aligned} \Psi_a &= a_1(t)f_1(\vec{r}) + a_2(t)f_2(\vec{r}), \\ \Psi_m &= b_1(t)c_1(\vec{r}) + b_2(t)c_2(\vec{r}). \end{aligned} \tag{5}$$

Here, we assume that dynamics of weakly coupled ANBEC can be described in four-mode approximation, and a_1, b_1 and a_2, b_2 describe time dependence of atomic and molecular condensate in different wells of the trap. The modal functions f_1, f_2, c_1, c_2 are defined as a ground state of a single isolated potential well [27]

$$\Psi_a = e^{-i\mu t} f(\vec{r}),\ \Psi_m = e^{-2i\mu t} c(\vec{r}), \tag{6}$$

and the numerical solution of the corresponding eigenvalue problem [10]

$$\begin{aligned} \hbar\mu f &= -\frac{\hbar^2}{2m}\Delta f + V(\vec{r})f + 2\lambda' f^* c + \lambda'_a |f^2| f, \\ 2\hbar\mu c &= -\frac{\hbar^2}{4m}\Delta c + V(\vec{r})c + \lambda' f^2. \end{aligned} \tag{7}$$

Another way is to use the variational approach with the Gaussian anzatz for the mode functions f_i, c_i [11]. While the algebraic constraints on the parameters are complicated, they can be solved numerically. If the modes are not significantly modified by the mean field effects then the harmonic oscillator wave functions can be used

$$\begin{aligned} f_{1,2} &= \left(\frac{1}{2\pi\Delta_1}\right)^{3/4} e^{-(x^2+y^2+z^2)/(4\Delta_1)}, \\ c_{1,2} &= \left(\frac{1}{2\pi\Delta_2}\right)^{3/4} e^{-(x^2+y^2+z^2)/(4\Delta_2)}, \end{aligned}$$

where $\Delta_1 = \hbar/2M\omega_1$, and $\Delta_2 = \hbar/4M\omega_2$. ω_1, ω_2 are frequencies of small oscillations in the trap for atomic and molecular condensates respectively.

Substituting this expansion into (2,3) and multiplying both sides on $f_i(\vec{r})$ or $c_i(\vec{r})$ and integrating over $\vec{r}$ we obtain the system of equations for the coupled modes

$$i\hbar\frac{da_1}{dt} = Wa_1 + Ka_2 + g|a_1|^2 a_1 + \chi a_1^* b_1 \tag{8}$$

$$i\hbar\frac{da_2}{dt} = Wa_2 + Ka_1 + g|a_2|^2 a_2 + \chi a_2^* b_2 \tag{9}$$

$$i\hbar\frac{db_1}{dt} = wb_1 + kb_2 + \frac{\chi}{2}a_1^2 \tag{10}$$

$$i\hbar\frac{db_2}{dt} = wb_2 + kb_1 + \frac{\chi}{2}a_2^2 \tag{11}$$

where

$$\begin{aligned} W &= -\frac{\hbar^2}{2M}\int f_1^* \Delta f_1 d\vec{r} + \int V_1 |\ f_1\ |^2 d\vec{r}, \\ w &= -\frac{\hbar^2}{4M}\int c_1^* \Delta c_1 d\vec{r} + \int V |\ c_1\ |^2 d\vec{r}, \\ K &= -\frac{i\hbar^2}{2M}\int f_1^* \Delta f_2 d\vec{r} + \int V f_1^* f_2 d\vec{r}, \\ k &= -\frac{i\hbar^2}{4M}\int c_1^* \Delta c_2 d\vec{r} + \int V c_1^* c_2 d\vec{r}, \\ \chi &= 2\lambda' \int f_1^{*2} c_1 d\vec{r}, \quad g = \lambda'_a \int |\ f_1\ |^4 d\vec{r}. \end{aligned} \tag{12}$$

Values of the parameters in the model are $K/\hbar = 1$ kHz, $\chi/\hbar = 20$ Hz, $g/\hbar$ is changed within (0−1) Hz.

Here, we should note that, as was shown in [18,19] in a typical parameters range one cannot use a few mode approximation to describe AMBEC. The four modes model that we will consider in principle is applicable at the very low temperatures or for small size systems, when we can detune from the excited modes [18]. The modern experiment requires the taking into account the multimode character of the problem. The latter predict the damping of oscillations between molecular and atomic components of the condensate. The position of the first oscillation is predicted by the mean field correctly. As we will interested by the macroscopic quantum self trapping (nonoscillating regimes) thresholds etc., our consideration in this part can be interesting for planning experiments too.

Before proceeding with the system (8–11) let us describe shortly the known results for the dynamics of AMBEC in a single well trap. In this case one neglects the coupling terms ($K = k = 0$), and the system (8–11) becomes

$$i\hbar\frac{da_1}{dt} = Wa_1 + g|a_1|^2 a_1 + \chi a_1^* b_1, \tag{13}$$

$$i\hbar\frac{db_1}{dt} = wb_1 + \frac{\chi}{2}a_1^2. \tag{14}$$

If the interatomic interaction is negligible, $g = 0$, then equations (13–14) coincide with the well known from nonlinear optics equations of second harmonic generation, and they have exact solution, which in terms of AMBEC describes the full conversion of atomic condensate to molecular one. The inclusion of the interatomic interaction leads [4,6] to the appearance of oscillation in the dynamics of population of atomic and molecular condensate, because of tunneling current between them. When the strength of interatomic nonlinear interaction becomes larger than some critical value the self-maintained population imbalance appears: macroscopic quantum self trapping effect in AMBEC [17].

There is another limiting case which was investigated in the literature, when one neglects the atom-molecular interaction ($\chi = \kappa = 0$) but puts the system to a double well trap. In this case we have the atomic Bose-Einstein condensate in double well trap [28–30] and equations (8–11)

become,

$$i\hbar\frac{da_1}{dt} = Wa_1 + Ka_2 + g|a_1|^2 a_1 \quad (15)$$

$$i\hbar\frac{da_2}{dt} = Wa_2 + Ka_1 + g|a_2|^2 a_2. \quad (16)$$

The system (15, 16) also show the appearance of oscillating current between coupled condensates below critical value of nonlinearity coefficient g, and macroscopic quantum self trapping effect when it is above critical value.

So, from a general point of view, it is possible to outline, that the system (8–11) describing weekly coupled AMBECs, consists of two subsystems both with nonlinear oscillations and macroscopic quantum tunneling, and one may expect that the dynamics generated by coupling such systems exhibits complicated behaviour.

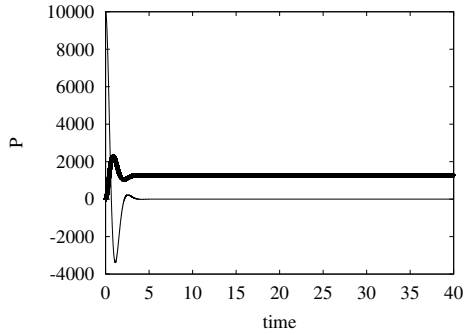

Fig. 1. The atom condensate (thin line) and molecular condensate (thick line) intermodal populations imbalance P_a and P_m as a function of normalized time τ for $E_1 = E_2 = \Lambda = \epsilon = 0$, $\chi_1 = 0.02$.

3 Analysis of the modes equations

In this section the results of numerical simulation of system (8–11) for different values of control parameters are presented and their physical meaning is discussed. Let us introduce the following dimensionless variable $\tau = tK/\hbar$. Then the problem of AMBEC's in a double-well time dependent traps can be described by the following four mode model — the extension of two-mode models considered for AMBEC condensate in single trap [4,5]

$$i\frac{da_1}{d\tau} = E_1 a_1 + a_2 + \Lambda|a_1|^2 a_1 + \chi_1 a_1^* b_1, \quad (17)$$

$$i\frac{da_2}{d\tau} = E_1 a_2 + a_1 + \Lambda|a_2|^2 a_2 + \chi_1 a_2^* b_2 \quad (18)$$

$$i\frac{db_1}{d\tau} = E_2 b_1 + \epsilon b_2 + \frac{\chi_1}{2} a_1^2, \quad (19)$$

$$i\frac{db_2}{d\tau} = E_2 b_2 + \epsilon b_1 + \frac{\chi_1}{2} a_2^2 \quad (20)$$

where $E_1 = W/K, E_2 = w/K, \epsilon = k/K, \Lambda = g/K, \chi_1 = \chi/K$.

Uniform amplitudes are $a_i = \sqrt{N_{i,a}}\exp(i\theta_{i,a})$, $b_i = \sqrt{N_{i,m}}\exp(i\theta_{i,m})$, $i = 1, 2$ determines the well in the trap. It is interesting to investigate the dynamics of the populations imbalance $P_a(t) = (N_{1,a} - N_{2,a}), P_m(t) = (N_{1,m} - N_{2,m})$. We can expect the localization of the populations in the separate modes in dependence of the value of nonlinearity parameters. In all calculations in this section the following initial conditions have been used: $a_1 = 100, a_2 = a_3 = a_4 = 0$.

Let us start the investigation for the case when we neglect the nonlinear interatomic interaction and coupling of molecular condensates in different traps. Molecular coupling should be much smaller than atomic coupling, because it depends on tunneling rates, and obviously the tunneling rates of molecules are much smaller than for atoms. The interatomic interaction may be tuned close to zero by external fields, so the investigation of this limiting case is also of physical interest. In Figure 1 we show the result of simulations. From these results one can deduce that in this case the atomic condensate, after some transient oscillations, fully converted to molecular one, which distributed in both molecular modes, and asymptotically, there is only molecular condensate. Then we have solved the system (17–20) with nonzero values of g and ϵ. In Figure 2a the example of simulation is presented when we include the small coupling of the molecular modes, which should exist in real a system. This coupling, even it is quite small, leads to the oscillations of population in all modes and the frequency of oscillations is proportional to the value of molecular modes coupling coefficient. So, there is the instability of solution presented in Figure 1, for small values of the molecular intermodal coupling ϵ. We have checked also (Fig. 2b), that the same type of instability appear when we take into account the nonlinear interatomic interaction. This also leads to oscillations in all modes of weakly coupled AMBEC, and the dynamics of oscillations looks quite irregular.

We now proceed with the solution of equations (17–20) with all terms included. The aim of this study was to check in details the influence of the strength of interatomic interaction g to the dynamics of AMBEC. In Figures 3a–3d we show the dynamics of atom condensate P_a and molecular condensate P_m population imbalance as a function normalized time τ, for the increasing values of g. From these figures we could note that nonlinearity effectively dump the conversion of atomic condensate to molecular one, and this fact is known also from two mode model of AMBEC, where the MQST effect exists. For small values of g atomic condensate could be converted to molecular one, but period of oscillations and its amplitude become variable. With increasing of the value of g, the conversation is fully deteriorated. Also, just as in the case of two weakly coupled atom condensates, there are localization phenomena of the atomic condensate in one of the traps. But, in the case of two weakly coupled AMBECs, there are no clear separation of two dynamical regimes, oscillations and localization, but instead one may observe the region in phase-space, in which oscillations disappear and localization appears through some irregular, chaotic oscillations. This phenomena could be explained

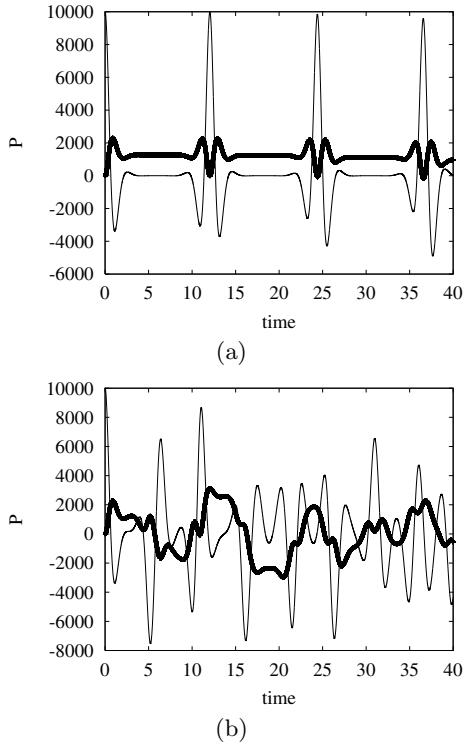

Fig. 2. (a) The atom condensate (thin line) and molecular condensate (thick line) intermodal populations imbalance P_a and P_m as a function of normalized time τ for $E_1 = E_2 = \Lambda = 0$, $\chi_1 = 0.02$, $\epsilon = 0.0001$; (b) for $E_1 = E_2 = 0$, $\chi_1 = 0.02$, $\epsilon = 0$, $\Lambda = 0.00001$.

qualitatively as a result of interaction of atomic and molecular subsystems of the condensate.

Let us consider the limiting case, when $E_2 b_i \approx -\alpha_1 {a_i}^2$. Then we have the system of equations,

$$ia_{1t} = (E_1 - 2\alpha^2/E_2)|a_1|^2 a_1 + K a_2,$$
$$ia_{2t} = (E_1 - 2\alpha^2/E_2)|a_2|^2 a_2 + K a_1. \qquad (21)$$

Introducing variables $a_1 = \sqrt{n_1}\exp(i\theta_1)$, $a_2 = \sqrt{n_2}\exp(i\theta_2)$ we obtain the system of equations

$$\Psi_t = U_1 n z - \frac{2K}{\sqrt{1-z^2}}\cos(\Psi),$$
$$z_t = 2K\sqrt{1-z^2}\sin(\Psi), \qquad (22)$$

where $\Psi = \theta_2 - \theta_1$, $z = (n_1 - n_2)/n, n = n_1 + n_2 = const$, $U_1 = E_1 - 2\alpha^2/E_2$. This is the system describing two weekly coupled Bose-Einstein condensates, and its dynamics both in classic and quantum cases is investigated in [28–30]. The Hamiltonian for this system is $H = \Lambda z^2/2 + \sqrt{1-z^2}\cos(\Psi)$. Let us introduce the parameter $\Lambda = U_1 n$. When $\Lambda = 5, \Psi(0) = 0, z(0) = 0.3$ we should have the oscillations with $\langle z\rangle = 0$ — so-called macroscopic quantum tunneling regime. When $\Lambda = 10, z(0) = 0.6, \Psi(0) = 0$ we should observe oscillations with the mean $\langle z\rangle \neq 0$ — so-called self-trapping regime. In derivation of equations (21) we have eliminated adiabatically the molecular subsystem, because the rate of

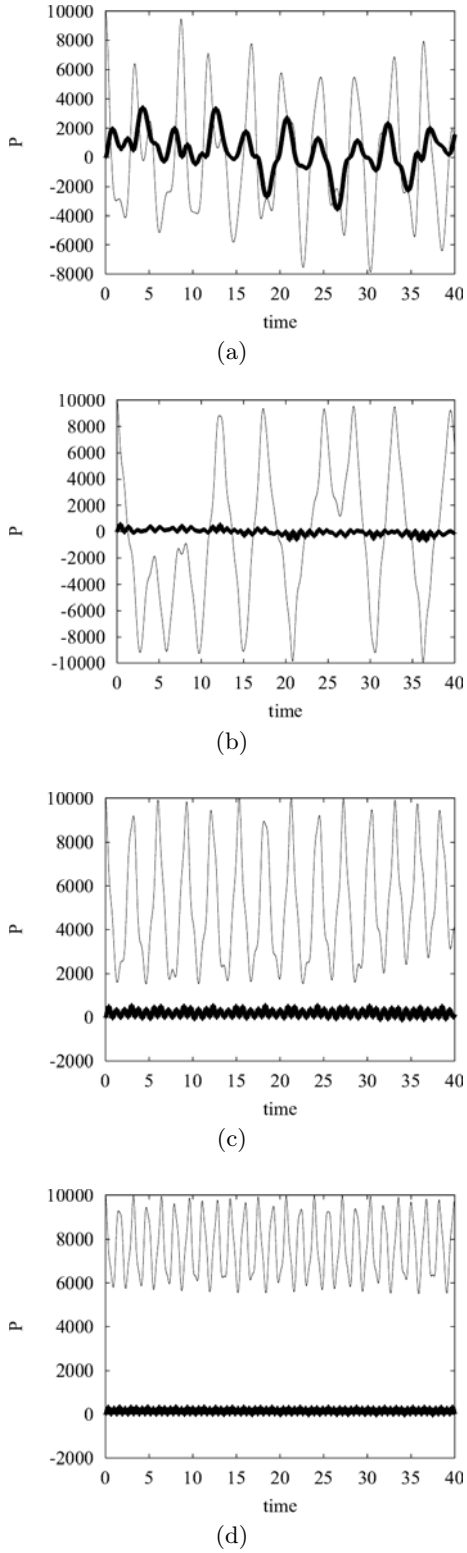

Fig. 3. (a) The atom condensate (thin line) and molecular condensate (thick line) intermodal populations imbalance P_a and P_m as a function of normalized time τ for $E_1 = E_2 = 0$, $\chi_1 = 0.02$, $\epsilon = 0.0001$, $\Lambda = 0.0001$; (b) for $E_1 = E_2 = 0$, $\chi_1 = 0.02$, $\epsilon = 0.0001$, $\Lambda = 0.000375$; (c) for $E_1 = E_2 = 0$, $\chi_1 = 0.02$, $\epsilon = 0.0001$, $\Lambda = 0.0004$; (d) for $E_1 = E_2 = 0$, $\chi_1 = 0.02$, $\epsilon = 0.0001$, $\Lambda = 0.0005$.

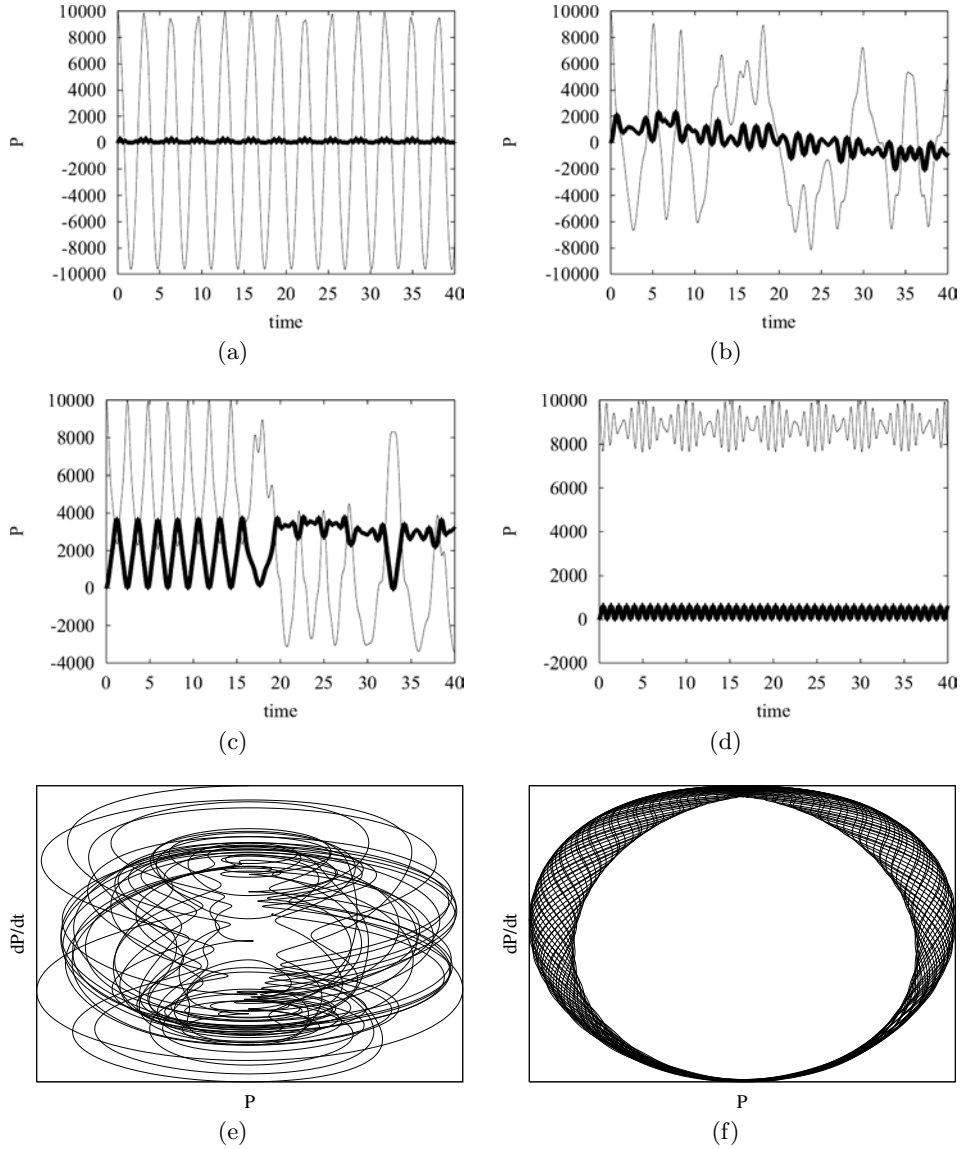

Fig. 4. (a) The atom condensate (thin line) and molecular condensate (thick line) intermodal populations imbalance P_a and P_m as a function of normilized time τ for $E_2 - E_1 = 10$, $\chi_1 = 0.02$, $\epsilon = 0.001$, $\Lambda = 0.00001$; (b) for $E_2 - E_1 = 10$, $\chi_1 = 0.02$, $\epsilon = 0.001$, $\Lambda = 0.00055$; (c) for $E_2 - E_1 = 10$, $\chi_1 = 0.02$, $\epsilon = 0.001$, $\Lambda = 0.00075$; (d) for $E_2 - E_1 = 10$, $\chi_1 = 0.02$, $\epsilon = 0.001$, $\Lambda = 0.0009$; (e) the phase portrait for atom condensate intermodal populations imbalance for the values of parameters the same as in (a); (f) the phase portrait for atom condensate intermodal populations imbalance for the values of parameters the same as in (c).

conversation is small in this case, as is the case in the analogous theory of second harmonic generation in nonlinear optics. It seems that the influence of the molecular condensate to dynamics of atomic one should be negligible. We have checked this assumption by numerical simulations of system (17–20).

In Figures 4a–4d we plot the population imbalance dynamics for atom and molecular condensates for different values of g, and for $E_1 - E_2 = 10$. For small values of nonlinearity the conversation to molecular condensate is really small, and its influence to the dynamics of atom condensate is negligible, so its dynamics described by equations (19) satisfactorily.

We can wait the essential influence even of a small fraction of the molecular component on the atomic condensate tunneling near the transition to a macroscopic self-trapping case. The critical value of the parameter α can be obtained from the requirement $H(0) \geq 1$. Then we find

$$\alpha^2 n_c = \frac{\left[E_1 z^2(0) - 2\left(1 + \sqrt{1 - z^2(0)}\cos\phi(0)\right) E_2\right]}{2z^2(0)}. \tag{23}$$

But near the separatrix, where MQST phenomena should appear, the influence of even small oscillations of molecular condensate leads again to irregular, chaotic oscillation in atomic condensate dynamics, and even more, the conversation rate to molecular subsystem could be enhanced, and its dynamics also could be chaotic. If the value of nonlinearity much more than critical value the influence of molecular condensate to the dynamics of atomic one becomes again negligible, and two mode approximation works well. In Figures 4e and 4f the phase portraits are plotted for the values of parameters corresponding to Figures 4a and 4c. The quasiperiodic dynamics for small values of nonlinearity, and chaotic motion for the values of nonlinearity near separatrix are clearly seen from these graphs.

So, in the nonresonant case the influence of molecular condensate could be noticeable near the bifurcation point of dynamical system, and at this point one should take into account oscillations of molecular condensate, even if they are small, because they may resonant with the oscillations of atom condensate. Recently periodic oscillations were observed in the atom number in one well [9]. For the analysis of the dynamics in two wells we needs in the extension of the resonance mean field theory developed in [13] for this case (taking into account the resonance in the pairing effect). It is subject of the future investigation.

4 Quantum dynamics of two weakly coupled AMBEC

The analysis which performed in the previous sections is not taking into account the influence of quantum fluctuations. For this we should going beyond of the four modes approximation. One of the possible ways is to derive the stochastic equations for the full quantum states. For one well case this procedure has been applied recently by [16]. Below we following to this procedure will consider quantum dynamics in double-well trap potential.

The Hamiltonian operator for the system under consideration, in the four mode approximation, can be written in the following form [17,27],

$$\hat{H} = \hat{H}_1 + \hat{H}_2 + (\hat{a}_1^+\hat{a}_2 + \hat{a}_2^+\hat{a}_1) + \epsilon(\hat{b}_1^+\hat{b}_2 + \hat{b}_2^+\hat{b}_1),$$

$$\hat{H}_i = E_1\hat{a_i}^+\hat{a_i} + E_2\hat{b_i}^+\hat{b_i} + \Lambda\hat{a_i}^{+2}\hat{a_i}^2 + \chi_1(\hat{b_i}^+\hat{a_i}^2 + \hat{b_i}\hat{a_i}^{+2}), \tag{24}$$

here $i = 1, 2$ determines the wells of the trap, $\hat{a_i}$, and $\hat{b_i}$ are the annihilation operators of atomic and molecular condensate respectively.

By application of standard techniques for the Hamiltonian (24) we obtain the master equation for the density matrix of the system,

$$\frac{\partial\hat{\rho}}{\partial t} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}]$$

which in positive P representation [34] can be converted to the Fokker-Planck equation for the quasiprobability distribution function $P(\alpha_i, \beta_i, \gamma_i, \delta_i)$

$$\begin{aligned}\frac{\partial P}{\partial z} = i\Big[&\frac{\partial}{\partial\alpha_1}(E_1\alpha_1 + 2\Lambda\beta_1\alpha_1^2 + \alpha_2 + \chi_1\gamma_1\beta_1)\\ &- \frac{\partial}{\partial\beta_1}(E_1\beta_1 + 2\Lambda\alpha_1\beta_1^2 + \beta_2 + \chi_1\delta_1\alpha_1)\\ &+ \frac{\partial}{\partial\alpha_2}(E_1\alpha_2 + 2\Lambda\beta_2\alpha_2^2 + \alpha_1 + \chi_1\gamma_2\beta_2)\\ &- \frac{\partial}{\partial\beta_2}(E_1\beta_2 + 2\Lambda\alpha_2\beta_2^2 + \beta_1 + \chi_1\delta_2\alpha_2)\\ &- \frac{\partial}{\partial\delta_1}(E_2\delta_1 + \chi_1\beta_1^2/2 + \epsilon\delta_2)\\ &+ \frac{\partial}{\partial\gamma_1}(E_2\gamma_1 + \chi_1\alpha_1^2/2 + \epsilon\gamma_2)\\ &- \frac{\partial}{\partial\delta_2}(E_2\alpha_2 + \chi_1\beta_2^2/2 + \epsilon\delta_1)\\ &+ \frac{\partial}{\partial\gamma_2}(E_2\gamma_2 + \chi_1\alpha_2^2/2 + \epsilon\gamma_1)\\ &+ \frac{1}{2}\frac{\partial^2}{\partial\alpha_1^2}(-2\Lambda\alpha_1^2 - \chi_1\gamma_1) + \frac{1}{2}\frac{\partial^2}{\partial\beta_1^2}(2\Lambda\beta_1^2 + \chi_1\delta_1)\\ &-\frac{1}{2}\frac{\partial^2}{\partial\alpha_2^2}(-2\Lambda\alpha_2^2 + \chi_1\gamma_2) + \frac{1}{2}\frac{\partial^2}{\partial\beta_2^2}(2\Lambda\beta_2^2 + \chi_1\delta_2)\Big]P.\end{aligned}$$

Using the Ito rules one can obtain from the Fokker-Planck equation the Langevin stochastic equations for the $\alpha_i, \beta_i, \gamma_i, \delta_i$ variables as

$$\begin{aligned}\frac{d\alpha_1}{d\tau} &= -i\left(E_1\alpha_1 + \Lambda\beta_1\alpha_1^2 + \chi_1\gamma_1\beta_1 + \alpha_2\right)\\ &\quad + \sqrt{-i(\Lambda\alpha_1^2 + \chi_1\gamma_1)}\eta_1(z), \qquad (25)\\ \frac{d\beta_1}{d\tau} &= i\left(E_1\beta_1 + \Lambda\beta_1^2\alpha_1 + \chi_1\delta_1\alpha_1 + \beta_2\right)\\ &\quad + \sqrt{i(\Lambda\beta_1^2 + \chi_1\delta_1)}\eta_2(z), \qquad (26)\\ \frac{d\alpha_2}{d\tau} &= -i\left(E_1\alpha_2 + \Lambda\beta_2\alpha_2^2 + \chi_1\gamma_2\beta_2 + \alpha_1\right)\\ &\quad + \sqrt{-i(\Lambda\alpha_2^2 + \chi_1\gamma_2)}\eta_3(z), \qquad (27)\\ \frac{d\beta_2}{d\tau} &= i\left(E_1\beta_2 + \Lambda\beta_2^2\alpha_2 + \beta_1 + \chi_1\delta_2\alpha_2 + \beta_1\right)\\ &\quad + \sqrt{i(\Lambda\beta_2^2 + \chi_1\delta_2)}\eta_4(z), \qquad (28)\\ \frac{d\gamma_1}{d\tau} &= -i\left(E_2\gamma_1 + \frac{\chi_1}{2}\alpha_1^2 + \epsilon\gamma_2\right), \qquad (29)\\ \frac{d\delta_1}{d\tau} &= i\left(E_2\delta_1 + \frac{\chi_1}{2}\beta_1^2 + \epsilon\delta_2\right), \qquad (30)\\ \frac{d\gamma_2}{d\tau} &= -i\left(E_2\gamma_2 + \frac{\chi_1}{2}\alpha_2^2 + \epsilon\gamma_1\right), \qquad (31)\\ \frac{d\delta_2}{d\tau} &= i\left(E_2\delta_2 + \frac{\chi_1}{2}\beta_2^2 + \epsilon\delta_1\right). \qquad (32)\end{aligned}$$

In equations (25–32) we use the same normalization as in equations (17–20). Here η_i are the independent real Langevin sources of noise with the following nonzero correlation functions:

$$\langle\eta_i^2\rangle = 1. \tag{33}$$

The Hamiltonian (24) takes into account only the coherent quantum evolution of the system, and derived stochastic equations just model the time evolution of density matrix via master equation. We do not consider in this paper the interaction with the reservoir and loss.

The initial conditions are deterministic for coherent initial fields and $\alpha_i(0) = \alpha_{c,i}$, $\beta_i(0) = \alpha^*_{c,i}$, $\gamma_i(0) = \alpha_{c,i}$, $\delta_i(0) = \alpha^*_{c,i}$, where $\mid \alpha_{c,i} \mid^2$, $\mid \beta_{c,i} \mid^2$ is the initial mean atom and molecules population in condensates. In positive P representation the quantum averages are found as moments of the P distribution function [34] that correspond to normally ordered expectation values. It should be noted that P-positive stochastic simulations method gives exact results for quantum system, within sampling error, but it have well known shortage [35], namely, for improper values of nonlinearity and calculation time instability may appear during numerical solution. So, we have to limit our calculations by small values of nonlinearity and evolution time, so to stop calculations before first instability appears.

Let us now discuss the results of our numerical simulations of the system of equations (25–32). The incident fields are assumed to be in the coherent states. The following physical quantities have been calculated. The quantum mean populations of atomic $N_{a,i}$ or molecular $N_{m,i}$ condensates in single modes $N_{a,i}(\tau) = \langle \hat{a}_i^\dagger \hat{a}_i \rangle$, $N_{m,i}(\tau) = \langle \hat{b}_i^\dagger \hat{b}_i \rangle$, and variances $\langle \Delta \hat{X}^2_{1,i} \rangle$, $\langle \Delta \hat{X}^2_{2,i} \rangle$ of the quadrature operators $\hat{X}_{1,i}$ and $\hat{X}_{2,i}$ of the single modes, where $\hat{X}_{1,i} = \frac{1}{2}(\hat{a}_i + \hat{a}_i^\dagger)$, $\hat{X}_{2,i} = -\frac{i}{2}(\hat{a}_i - \hat{a}_i^\dagger)$, $\hat{X}_{3,i} = \frac{1}{2}(\hat{b}_i + \hat{b}_i^\dagger)$, $\hat{X}_{4,i} = -\frac{i}{2}(\hat{b}_i - \hat{b}_i^\dagger)$.

In Figure 5 we plot the evolution of population versus time in the first mode for the values of parameters the same as in Figure 1, initial conditions are the coherent state in first atomic mode $\alpha_{c,1} = 100$, and vacuum state in the rest $\alpha_{c,2} = \beta_{c,1} = \beta_{c,2} = 0$. For solution of the Gross-Pitaevskii four-mode equations we have observed in this case, see Figure 1, the full conversion of atomic condensate to molecular, without any oscillations in final state. When we solve the quantum equations and calculate the mean values, we observe oscillations in all modes, and obviously this is the result of influence of quantum fluctuations to unstable system. It is necessary to note that similar phenomena was first noted in recent papers [16,36] for the case of second harmonic generation and single AMBEC. So, in quantum treatment, quantum noise induces macroscopic revivals in AMBEC dynamics. Also, from Figure 5 it is difficult to make some conclusion about regularity of the observed oscillations, since the running time of simulations is quite short, but at least initially the amplitudes of excited modes dynamics do not look as strictly regular.

We also calculated the mean-field intensities for values of parameters, that were presented for classical case. Unlike the case when $g = \epsilon = 0$, in all our calculations when we put nonzero values for g and ϵ the behavior were close to that found classically.

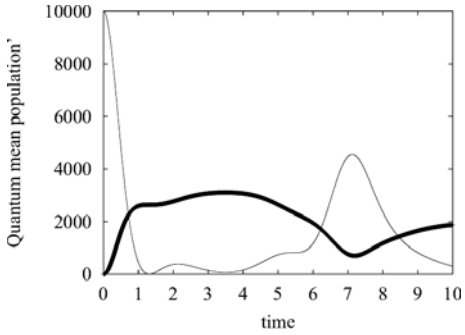

Fig. 5. The atom condensate (thin lines) and molecular condensate (thick lines) quantum mean populations $N_{a,i}$ and $N_{m,i}$ as a function of normalized time τ for $E_1 = E_2 = 10$, $\chi_1 = 0.02$, $\epsilon = \Lambda = 0$.

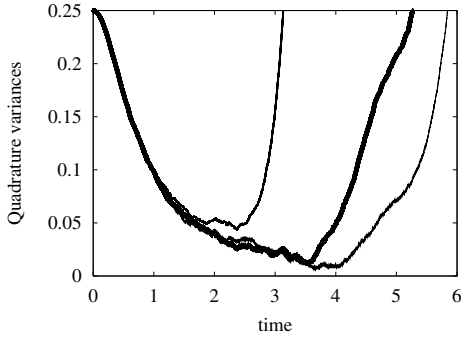

Fig. 6. The first mode atom condensate quadrature variances $X_{1,1}$ as a function of normalized time τ for $E_1 = E_2 = 0$, $\chi_1 = 0.02$, $\epsilon = 0.001$, and different Λ. $\Lambda = 0$, $\Lambda = 0.000001$, $\Lambda = 0.00001$.

Now we proceed with the calculations of the quadrature variances of single modes from which we can determine the possibility of squeezing for weakly coupled AMBECs. The shot noise level is 0.25. In Figure 6 the results of our calculations in positive P are presented. The nonclassical behavior is clearly seen. The transient, initial squeezing is observed for all calculated values of the nonlinear parameter. In Figure 6 we show the results for the first atomic channel where the value of squeezing are maximal, the evolution of quadrature variances are actually very similar for all channels. The maximal squeezing is obtained for zero value of interatomic interaction g. It is interesting to note, that the second mode, being initially in vacuum state, also have quadrature squeezing, so linear coupling leads to the correlations in quantum-statistics also. And it can be immediately seen that value of obtainable squeezing suppressed when value of nonlinear interatomic interaction g increased, and monotonically goes to zero with increasing of g. We do not observe some critical value for g, when the squeezing become zero. In longer times, the variances become to exceeds the shot noise level.

5 Conclusions

In summary, we have studied the tunneling dynamics of atomic-molecular Bose-Einstein condensates in double-well trap, and have shown that the tunneling rates of the atoms and molecules depend on not only the tunnel coupling between the atomic condensate and the molecular condensate and tunnel coupling between condensates in different wells of the trap, but also the inter-atomic nonlinear interactions and the initial number of atoms in these condensates. Especially, we have shown that the tunnel coupling and the inter-atomic nonlinear interactions strongly affect the tunneling of atomic pairs in the regime of weak tunnel coupling when the atomic condensate is in a coherent and the molecular condensate in the vacuum state. This implies that the tunneling of atomic pairs between the atomic condensate and the molecular condensate and between wells of the trap, can be manipulated and controlled by varying the tunnel coupling and/or inter-atomic nonlinear interaction strengths. We have revealed the existence of the chaotic dynamics and MQST between the atomic condensate and the molecular condensate. The MQST is a kind of nonlinear effects which vanishes in the absence of the inter-atomic nonlinear interactions. We have shown that in some range of parameters, quantum fluctuations can strongly affect the quantum means dynamics and contradict with the results obtained from solution of classical equations. The possibility of quadrature squeezing is shown for coupled AMBEC. Namely the second mode, being initially in vacuum state, also have quadrature squeezing, so linear tunnel coupling between two AMBEC leads to the correlations in quantum-statistics.

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References

1. M.H. Anderson, J.R. Ensher, M.R. Matthews, C.E. Wieman, E.A. Cornell, Science **269**, 198 (1995)
2. K.B. Davis, M.-O. Mewes, M.R. Andrews, N.J. van Druten, D.S. Durfee, D.M. Kurn, W. Ketterle, Phys. Rev. Lett. **75**, 3969 (1995)
3. C.C. Bradley, C.A. Sackett, R.G. Hulet, Phys. Rev. Lett. **75**, 1687 (1995); **78**, 985 (1997)
4. E. Timmermans, P. Tommasini, R. Côte, M. Hussein, A. Kerman, Phys. Rev. Lett. **83**, 2691 (1999); E. Timmermans, P. Tommasini, M. Hussein, A. Kerman, Phys. Rep. **315**, 199 (1999)
5. F.A. van Abeelen, B.J. Verhaar, Phys. Rev. Lett. **83**, 1550 (1999)
6. P. Drummond, K.V. Kheruntsyan, H. He, Phys. Rev. Lett. **81**, 3055 (1999); D.J. Heinzen, Roahn Wynar, P.D. Drummond, K.V. Kheruntsyan, Phys. Rev. Lett. **84**, 5029 (2000)
7. R. Duine, H.T.C. Stoof, Phys. Rep. **396**, 115 (2004)
8. R. Wynar, R.S. Freeland, D.J. Han, C. Ryu, D.J. Heinzen, Science **287**, 1016 (2000)
9. E. Donley, N. Claussen, S. Thomson, C. Wieman, Nature **417**, 529 (2002)
10. B.J. Cusack, T.J. Alexander, E.A. Ostrovskaya, Y.S. Kivshar, Phys. Rev. A **65**, 013609 (2002)
11. V.V. Steblina, Yu.S. Kivshar, M. Lisak, B. Malomed, Opt. Commun. **118**, 345 (1995)
12. M. Mackie, R. Kowalski, J. Javanainen, Phys. Rev. Lett. **84**, 5029 (2000)
13. S.J.J.M.F. Kokkelmans, H.M.J. Vissers, B.J. Verhaar, Phys. Rev. A **63**, 031601(R) (2001)
14. U.V. Poulsen, K. Molmer, Phys. Rev. **63**, 023604 (2001)
15. J.J. Hope, M.K. Olsen, L.I. Plimak, Phys. Rev. A **63**, 043603 (2001)
16. J.J. Hope, M.K. Olsen, Phys. Rev. Lett. **86**, 3220 (2001)
17. Bambi Hu, Le Man-Kuang, Phys. Rev. A **62**, 023610 (2000)
18. K. Goral, M. Gajda, K. Rzazewski, Phys. Rev. Lett. **86**, 1397 (2001)
19. M. Holland, J. Park, R. Walser, Phys. Rev. Lett. **86**, 1915 (2001)
20. M.R. Andrews, C.G. Townsend, H.-J. Miesner, D.S. Durfee, D.M. Kurn, W. Ketterle, Science **275**, 637 (1997)
21. J. Javanainen, Phys. Rev. Lett. **57**, 3164 (1986)
22. F. Dalfovo, L. Pitaevski, S. Stringari, Phys. Rev. A **54**, 4213 (1996)
23. A. Barone, B. Paterno, *Physics and Applications of the Josephson Effect* (Wiley, New York, 1982)
24. S.M. Jensen, IEEE J. Quant. Electron. **18**, 1580 (1982); A.A. Maier, Kvant. Elektron. **9**, 2296 (1982)
25. D. Marcuse, *Theory of Optical Waveguides* (Wiley, 1975)
26. F.Kh. Abdullaev, R.M. Abrarov, S.A. Darmanyan, Opt. Lett. **14**, 135 (1989)
27. G.J. Milburn, J. Corney, E.M. Wright, D.F. Walls, Phys. Rev. A **55**, 4318 (1997)
28. D. Gordon, C.M. Savage, Phys. Rev. A **59**, 4623 (1999); M.J.A. Imamo$\bar{g}$lu, M. Lewenstein, L. You, Phys. Rev. Lett. **78**, 2511 (1997)
29. A. Smerzi, S. Giovanazzi, S.R. Shenoy, Phys. Rev. Lett. **79**, 4950 (1997); S. Raghavan, A. Smerzi, S. Fantoni, S.R. Shenoy, Phys. Rev. A **59**, 620 (1999)
30. I. Zapata, F. Sols, A. Leggett, Phys. Rev. A **57**, R28 (1998)
31. F.Kh. Abdullaev, R.A. Kraenkel, Phys. Rev. A **62**, 023613 (2000)
32. G. Assanto, G. Stegeman, M. Shiek-Bahae, E. Van Stryland, Appl. Phys. Lett. **62**, 1323 (1993); G. Assanto, A. Laureti-Palma, C. Sibilia, M. Bertolotti, Opt. Commun. **110**, 599 (1994)
33. F.Kh. Abdullaev, R.A. Kraenkel, Phys. Lett. A **272**, 395 (2000)
34. C.W. Gardiner, *Quantum Noise* (Springer-Verlag, Berlin, 1991)
35. A. Gilchrist, P.D. Drummond, C.W. Gardiner, Phys. Rev. A **55**, 3014 (1997)
36. M.A. Olsen, R.J. Horowicz, L.J. Plimak, N. Treps, C. Fabre, Phys. Rev. A **61**, 021803(R) (2000)
37. S. Kokkelmans, M. Holland, Phys. Rev. Lett. **89**, 18040 (2002)